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Explicit soliton–black hole correspondence for static configurations

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Abstract
We construct an explicit map that transforms static, generalized sine-Gordon metrics to black hole type metrics. This, in particular, provides for a further description of the Cadoni correspondence (which extends the Gegenberg–Kunstatter correspondence) of soliton solutions and extremal black hole solutions in 2D dilaton gravity.

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1. Introduction

An interesting, intriguing connection between Euclidean N-soliton sine-Gordon solutions and Lorentzian black hole solutions in Jackiw–Teitelboim dilaton gravity has been established by Gegenberg and Kunstatter [4–6]. In case $N = 1$, a concrete transformation was constructed that explicates this connection [9] (see also [10, 11]). The construction of such a transformation in general seems to be a difficult problem as it involves, in particular, finding explicit solutions of a system of dilaton field equations. Recently, these field equations have been solved for a kink–antikink soliton (similar solutions were found in [5]), and thus an explicit transformation has been also constructed in this case [2] that further implements the work in [4, 5].

We present a transformation Ψ that takes any generalized static sine-Gordon type metric to a black hole type metric. In particular, we present additional solutions of the dilaton field equations (even in the non-static case) and a further description of the Cadoni correspondence [3] between extremal black holes and generalized solitons—generalized sine-Gordon solutions that do not give rise to constant curvature spacetimes, as in the Gegenberg–Kunstatter discussion.

We dedicate this paper to the memory of Professor Melvyn Berger—friend, and outstanding scholar in nonlinear phenomenon.

2. Field equations for 2D dilaton gravity

Given a potential function $V(r)$ and $l > 0$, we consider the general two-dimensional dilaton gravity theory with action integral

$$I(\tau, g) = \frac{1}{2G}\int d^2x\sqrt{-g}\left(\tau R(g) + \frac{V\circ\tau}{l^2}\right), \tag{1}$$

for which the equations of motion are

$$R(g) + \frac{V'\circ\tau}{l^2} = 0 \tag{2}$$

$$\nabla_\mu\nabla_\nu\tau + \frac{1}{2l^2}g_{\mu\nu}(V\circ\tau) = 0 \tag{3}$$

for the dilaton field $\tau(T, r)$ and metric g with scalar curvature $R = R(g)$. For $\tau(T, r) = \frac{r}{l}$, for example, and a constant C one has the well-known solution [1, 7]

$$ds^2 = -\left[-J\left(\frac{r}{l}\right) - C\right]dT^2 + \left[-J\left(\frac{r}{l}\right) - C\right]^{-1}dr^2 \tag{4}$$

for $J'(r) = V(r)$. Actually, $C = -l^2|\nabla\tau|^2 - J\circ\tau$, and $\frac{C}{2l}$ can be interpreted as the energy of the solution. For spherically symmetric gravity, for example, with $V(r) = -\frac{\gamma}{\sqrt{r}}, \gamma > 0$, one takes $l =$ the Planck length l_P.

For g given by

$$ds^2 = \cos^2\frac{u(x,t)}{2}\,dx^2 - \sin^2\frac{u(x,t)}{2}\,dt^2 \tag{5}$$

and for $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial t^2}$,

$$R = \frac{2\Delta u}{\sin u} \tag{6}$$

by our sign convention for the scalar curvature, which is opposite the sign of R in [3, 5], and the equations in (3) are

$$\begin{aligned}
&\tau_{xt} + \frac{1}{2}\tan\left(\frac{u}{2}\right)u_t\tau_x - \frac{1}{2}\cot\left(\frac{u}{2}\right)u_x\tau_t = 0\\
&\tau_{xx} + \frac{1}{2}\tan\left(\frac{u}{2}\right)u_x\tau_x + \frac{1}{2}\cot\left(\frac{u}{2}\right)u_t\tau_t + \frac{1}{2l^2}\cos^2\left(\frac{u}{2}\right)(V\circ\tau) = 0\\
&\tau_{tt} - \frac{1}{2}\tan\left(\frac{u}{2}\right)u_x\tau_x - \frac{1}{2}\cot\left(\frac{u}{2}\right)u_t\tau_t - \frac{1}{2l^2}\sin^2\left(\frac{u}{2}\right)(V\circ\tau) = 0.
\end{aligned} \tag{7}$$

By equations (2), (6) and by addition of the second and third equations in system (7) one obtains

$$\Delta u = -\frac{1}{2l^2}(V'\circ\tau)\sin u \qquad \Delta\tau = -\frac{1}{2l^2}(V\circ\tau)\cos u, \tag{8}$$

the first equation being a generalized sine-Gordon equation. Following Cadoni [3], we shall be interested in static field solutions $u(x,t) = u(x)$, $\tau(x,t) = \tau(x)$ in which case (8) reduces to the system

$$u''(x) = -\frac{1}{2l^2}V'(\tau(x))\sin u(x) \qquad \tau''(x) = -\frac{1}{2l^2}V(\tau(x))\cos u(x), \tag{9}$$

which has first integrals

$$u'(x) = -\frac{A}{l}V(\tau(x)) \qquad \tau'(x) = \frac{1}{2lA}\sin u(x), \tag{10}$$

for any constant $A \neq 0$. Note that by (10) one easily deduces that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[A^2 J(\tau(x)) + \sin^2\frac{u(x)}{2}\right] = 0 \tag{11}$$

for $J'(x) = V(x)$, which gives the conservation law

$$A^2 J(\tau(x)) + \sin^2\frac{u(x)}{2} = \text{a constant}, \tag{12}$$

which we express as

$$\sin^2\frac{u(x)}{2} = -A^2[J(\tau(x)) + C] \tag{13}$$

for a constant C. Using the second equation in system (10) and equation (13) one can also deduce that

$$\tau'(x)^2 = -\frac{1}{l^2}[J(\tau(x)) + C]\{1 + A^2[J(\tau(x)) + C]\}. \tag{14}$$

Equations (13), (14), which we have deduced by a conservation law, compare with equations (16), (17) in [3] where $-K$, $-V$ and Φ are our J, V and τ, respectively.

3. The transformations Ψ

In the static case under consideration we write the metric in (5) as

$$\mathrm{d}s^2_{\mathrm{sol}} = \cos^2\frac{u(x)}{2}\,\mathrm{d}x^2 - \sin^2\frac{u(x)}{2}\,\mathrm{d}t^2, \tag{15}$$

where the subscript 'sol' suggests the word soliton—given the sine-Gordon equation in (8). We look for an explicit map $\Psi = (\psi_1, \psi_2)$ and its inverse $\Theta = (\theta_1, \theta_2)$ such that under the change of variables $x = \psi_1(T, r)$, $t = \psi_2(T, r)$, the metric $\mathrm{d}s^2_{\mathrm{sol}}$ in (15) is transformed to the metric

$$\mathrm{d}s^2_{\mathrm{bh}} = -\left[-J\left(\frac{r}{l}\right) - C\right]\mathrm{d}T^2 + \left[-J\left(\frac{r}{l}\right) - C\right]^{-1}\mathrm{d}r^2 \tag{16}$$

in (4) for the C in equations (13), (14), where the subscript 'bh' suggests now some kind of generic extremal black hole. Conversely, under the change of variables $T = \theta_1(x, t)$, $r = \theta_2(x, t)$, $\mathrm{d}s^2_{\mathrm{bh}} \longrightarrow \mathrm{d}s^2_{\mathrm{sol}}$. It turns out that, in contrast to the more difficult non-static case, Ψ and Θ can be chosen to assume the following somewhat simple form.

Theorem 1. *Let $u(x)$ and $\tau(x)$ satisfy the system of first integrals (10). Then the following maps provide transformations between the soliton metric (15) and the extremal black hole metric (16):*

$$\begin{aligned} \psi_1(T, r) &= \tau^{-1}\left(\frac{r}{l}\right), & \psi_2(T, r) &= \frac{T - \theta_0}{A} \\ \theta_1(x, t) &= At + \theta_0, & \theta_2(x, t) &= l\tau(x), \end{aligned} \tag{17}$$

where τ^{-1} is the inverse function of τ and θ_0 is any constant.

Proof. Note first that, in general, for any change of variables $x = \psi_1(T, r)$, $t = \psi_2(T, r)$ the metric $\mathrm{d}s^2_{\mathrm{sol}}$ in equation (15) goes to

$$\mathrm{d}s^2 = \left[\left(\frac{\partial\psi_1}{\partial T}\right)^2 \cos^2\left(\frac{u\circ\Psi}{2}\right) - \left(\frac{\partial\psi_2}{\partial T}\right)^2 \sin^2\left(\frac{u\circ\Psi}{2}\right)\right]\mathrm{d}T^2$$
$$+2\left[\frac{\partial\psi_1}{\partial T}\frac{\partial\psi_1}{\partial r}\cos^2\left(\frac{u\circ\Psi}{2}\right) - \frac{\partial\psi_2}{\partial T}\frac{\partial\psi_2}{\partial r}\sin^2\left(\frac{u\circ\Psi}{2}\right)\right]\mathrm{d}T\,\mathrm{d}r$$
$$+\left[\left(\frac{\partial\psi_1}{\partial r}\right)^2 \cos^2\left(\frac{u\circ\Psi}{2}\right) - \left(\frac{\partial\psi_2}{\partial r}\right)^2 \sin^2\left(\frac{u\circ\Psi}{2}\right)\right]\mathrm{d}r^2,$$

for $\Psi = (\psi_1, \psi_2)$. If we wish to have $\mathrm{d}s^2 = \mathrm{d}s^2_{\mathrm{bh}}$ in equation (16), for C given in equation (13), we must therefore have:

$$\left(\frac{\partial\psi_1}{\partial T}\right)^2 \cos^2\left(\frac{u\circ\Psi}{2}\right) - \left(\frac{\partial\psi_2}{\partial T}\right)^2 \sin^2\left(\frac{u\circ\Psi}{2}\right) \stackrel{a}{=} J\left(\frac{r}{l}\right) + C$$
$$\frac{\partial\psi_1}{\partial T}\frac{\partial\psi_1}{\partial r}\cos^2\left(\frac{u\circ\Psi}{2}\right) - \frac{\partial\psi_2}{\partial T}\frac{\partial\psi_2}{\partial r}\sin^2\left(\frac{u\circ\Psi}{2}\right) \stackrel{b}{=} 0$$
$$\left(\frac{\partial\psi_1}{\partial r}\right)^2 \cos^2\left(\frac{u\circ\Psi}{2}\right) - \left(\frac{\partial\psi_2}{\partial r}\right)^2 \sin^2\left(\frac{u\circ\Psi}{2}\right) \stackrel{c}{=} \left[-J\left(\frac{r}{l}\right) - C\right]^{-1}.$$

Since we are focusing on the static case $u(x,t) = u(x)$, $\tau(x,t) = \tau(x)$ we have $u\circ\Psi = u\circ\psi_1$, $\tau\circ\Psi = \tau\circ\psi_1$. Also, by equation (13), $\cos^2\left(\frac{u\circ\psi_1}{2}\right) = 1 - \sin^2\left(\frac{u\circ\psi_1}{2}\right) = 1 + A^2[J\circ\tau\circ\psi_1 + C]$, which means that finding Ψ reduces to solving the system

$$\left(\frac{\partial\psi_1}{\partial T}\right)^2 \{1 + A^2[J\circ\tau\circ\psi_1 + C]\} - \left(\frac{\partial\psi_2}{\partial T}\right)^2 \sin^2\left(\frac{u\circ\psi_1}{2}\right) \stackrel{a'}{=} J\left(\frac{r}{l}\right) + C$$
$$\frac{\partial\psi_1}{\partial T}\frac{\partial\psi_1}{\partial r}\{1 + A^2[J\circ\tau\circ\psi_1 + C]\} - \frac{\partial\psi_2}{\partial T}\frac{\partial\psi_2}{\partial r}\sin^2\left(\frac{u\circ\psi_1}{2}\right) \stackrel{b'}{=} 0$$
$$\left(\frac{\partial\psi_1}{\partial r}\right)^2 \{1 + A^2[J\circ\tau\circ\psi_1 + C]\} - \left(\frac{\partial\psi_2}{\partial r}\right)^2 \sin^2\left(\frac{u\circ\psi_1}{2}\right) \stackrel{c'}{=} \left[-J\left(\frac{r}{l}\right) - C\right]^{-1}.$$

Clearly equation b' holds for $\frac{\partial\psi_1}{\partial T} \stackrel{d}{=} 0$, $\frac{\partial\psi_2}{\partial r} \stackrel{e}{=} 0$, in which case we write equation a' as

$$-\left(\frac{\partial\psi_2}{\partial T}\right)^2 \sin^2\left(\frac{u\circ\psi_1}{2}\right) = J\left(\tau\left(\tau^{-1}\left(\frac{r}{l}\right)\right)\right) + C = \frac{-1}{A^2}\left[\sin^2\left(\frac{u\left(\tau^{-1}\left(\frac{r}{l}\right)\right)}{2}\right)\right]$$

(by (13)), which suggests that we should try $\frac{\partial\psi_2}{\partial T} = \frac{1}{A}$ and $\psi_1(T,r) = \tau^{-1}\left(\frac{r}{l}\right)$. Equation e gives $\psi_2(T,r) = \frac{T}{A}+$ some constant. This shows how one arrives at the first transformation in equation (17), though one needs to check, conversely, that both formulae in (17) indeed satisfy equations a', b' and c' (which is done directly). One can derive the Θ equation similarly, but an easier route is to check indeed that $\Theta = (\theta_1, \theta_2)$ is the inverse of Ψ. □

Since $u(x)$ and $\tau(x)$ solve system (10), they also solve system (9). The verification that the transformations in (17) indeed do work relies heavily on the equations (13), (14), which as we have seen are implied by the equations in (10). The reader should not be misled by the simple appearance of the expressions of Ψ and Θ in the theorem; examples will demonstrate the complexity of τ^{-1}.

As a simple, but important example, choose $V(x) = -2x$, which provides for the Jackiw–Teitelboim model. Choose

$$u(x) = \pm 4\arctan \mathrm{e}^{\frac{x-x_0}{l}} \qquad \tau(x) = \pm\mathrm{sech}\left(\frac{x-x_0}{l}\right) \tag{18}$$

which solve system (10) for $A = 1$ (and which therefore solve system (9) and the sine-Gordon equation $u''(x) = \frac{1}{l^2}\sin u(x)$). Equation (13) then holds if and only if $C = 0$. By equation (17), the transformations of variables

$$\begin{aligned} x &= \psi_1(T, r) = x_0 + l\log\left[\frac{l + \sqrt{l^2 - r^2}}{\pm r}\right] \qquad & t &= \psi_2(T, r) = T, \\ T &= \theta_1(x, t) = t, & r &= \theta_2(x, t) = \pm l \,\mathrm{sech}\left(\frac{x - x_0}{l}\right) \end{aligned} \tag{19}$$

(where we choose $\theta_0 = 0$) take the soliton metric $\mathrm{d}s^2_{\mathrm{sol}}$ in (15) (for $u(x)$ given in (18)) to the extremal black hole metric $\mathrm{d}s^2_{\mathrm{bh}} = -\frac{r^2}{l^2}\,\mathrm{d}T^2 + \frac{l^2}{r^2}\,\mathrm{d}r^2$ and conversely $\mathrm{d}s^2_{\mathrm{bh}} \longrightarrow \mathrm{d}s^2_{\mathrm{sol}}$ (via Θ); here $J(x) = -x^2$. These transformations Ψ and its inverse Θ in (19) can be seen as implementing the Cadoni correspondence for the present example. Note a minor typing error in equation (31) of [3]: there one should have $\Phi^{-1} = \pm\cosh(\lambda(x - x_0))$ (instead of $\Phi^{-1} = \cosh(\lambda(x - x_0))$, as the minus sign is needed for the minus in (18)).

As pointed out in [3], a complete correspondence between 2D spacetime structures of the dilaton gravity theory and solutions in the generalized sine-Gordon field theory requires a consideration of metric (5) and of the metric

$$\mathrm{d}s^2 = \cosh^2\frac{u(x,t)}{2}\,\mathrm{d}x^2 - \sinh^2\frac{u(x,t)}{2}\,\mathrm{d}t^2 \tag{20}$$

as well, for the sinh-Gordon model. Here for $\Box = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}$, $R = \frac{2\Box u}{\sinh u}$ and thus equation (2) is now

$$\Box u + \frac{1}{2l^2}(V'\circ\tau)\sinh u = 0 \tag{21}$$

and the system in (3) becomes

$$\begin{aligned} &\tau_{xt} - \frac{1}{2}\tanh\left(\frac{u}{2}\right)u_t\tau_x - \frac{1}{2}\coth\left(\frac{u}{2}\right)u_x\tau_t = 0 \\ &\tau_{xx} - \frac{1}{2}\tanh\left(\frac{u}{2}\right)u_x\tau_x - \frac{1}{2}\coth\left(\frac{u}{2}\right)u_t\tau_t + \frac{1}{2l^2}\cosh^2\left(\frac{u}{2}\right)(V\circ\tau) = 0 \\ &\tau_{tt} - \frac{1}{2}\tanh\left(\frac{u}{2}\right)u_x\tau_x - \frac{1}{2}\coth\left(\frac{u}{2}\right)u_t\tau_t - \frac{1}{2l^2}\sinh^2\left(\frac{u}{2}\right)(V\circ\tau) = 0. \end{aligned} \tag{22}$$

From (21) and (22) one obtains in the static case the system

$$u''(x) = -\frac{1}{2l^2}V'(\tau(x))\sinh u(x) \qquad \tau''(x) = -\frac{1}{2l^2}V(\tau(x))\cosh u(x) \tag{23}$$

with first integrals

$$u'(x) = -\frac{A}{l}V(\tau(x)) \qquad \tau'(x) = \frac{1}{2Al}\sinh u(x), \tag{24}$$

that compares with system (10). Equations (13), (14) are replaced by

$$\begin{aligned} \sinh^2\frac{u(x)}{2} &= -A^2[J(\tau(x)) + C] \\ \tau'(x)^2 &= -\frac{1}{l^2}[J(\tau(x)) + C]\cdot\{1 - A^2[J(\tau(x)) + C]\} \end{aligned} \tag{25}$$

for a suitable constant C. For this C, and for $u(x)$, $\tau(x)$, that solve (24) (hence $u(x)$, $\tau(x)$ also solve equation (23)) one can check that for the metric $\mathrm{d}s^2$ in (20), $\mathrm{d}s^2 \longrightarrow \mathrm{d}s^2_{\mathrm{bh}}$ in (16), under the change of variables $(x, t) \longrightarrow \Psi(T, r) = (\psi_1(T, r), \psi_2(T, r))$, where

$$\begin{aligned} \psi_1(T,r) &= \tau^{-1}\left(\frac{r}{l}\right), \qquad & \psi_2(T,r) &= \frac{T-\theta_0}{A} \\ \theta_1(x,t) &= At+\theta_0, \qquad & \theta_2(x,t) &= l\tau(x); \end{aligned} \tag{26}$$

here $\Theta = (\theta_1, \theta_2) = \Psi^{-1}$. Thus Ψ, Θ here have the same form as the Ψ, Θ in (17).

4. The sinh-Φ model and other examples

Another model is defined by the potential $V(x) = -\sinh(2x)$. System (10) is solved by

$$\begin{aligned} u(x) &= \pi + 2\arctan\left[\sqrt{2}\sinh\left(\frac{x-x_0}{l}\right)\right] \\ \tau(x) &= \operatorname{arctanh}\left[\frac{1}{\sqrt{2}}\operatorname{sech}\left(\frac{x-x_0}{l}\right)\right] \end{aligned} \tag{27}$$

for $A = 1$, and equation (13) holds for $C = \frac{1}{2}$. $J(x) = -\frac{1}{2}\cosh(2x)$. The extremal black hole solution ds^2_{bh} corresponding to solution (27), in the Cadoni correspondence, is given by equation (16):

$$\begin{aligned} ds^2_{bh} &= -\left[\frac{1}{2}\cosh\left(\frac{2r}{l}\right) - \frac{1}{2}\right] dT^2 + \left[\frac{1}{2}\cosh\left(\frac{2r}{l}\right) - \frac{1}{2}\right]^{-1} dr^2 \\ &= -\sinh^2\left(\frac{r}{l}\right) dT^2 + \sinh^{-2}\left(\frac{r}{l}\right) dr^2, \end{aligned} \tag{28}$$

as in equation (40) of [3]. A change of variables $x = \psi_1(T,r), t = \psi_2(T,r)$ that takes ds^2_{sol} in (15) (for $u(x)$ in (27)) directly to ds^2_{bh} in (28) is given in (17), where we note that $\tau^{-1}(x) = x_0 + l\operatorname{arcsech}(\sqrt{2}\tanh(x))$ for $\tau(x)$ in (27):

$$\psi_1(T,r) = x_0 + l\log\left[\frac{1+\sqrt{1-2\tanh^2\frac{r}{l}}}{\sqrt{2}\tanh\frac{r}{l}}\right] \qquad \psi_2(T,r) = T, \tag{29}$$

where again we take $\theta_0 = 0$.

Going back to the Jackiw–Teitelboim model with $V(x) = -2x$, one can obtain another solution $u(x), \tau(x)$ of the field equations (9)—one of independent interest that involves the Jacobi elliptic functions sn, cn and dn [8]. For this, given $A \neq 0$ (as in system (10)) and a constant $E > 0$, define

$$\begin{aligned} B &= \frac{A^2E}{4l^2} \qquad K = 1 + \frac{2}{A^2E} \qquad \alpha = \sqrt{2K^2 + 2K\sqrt{K^2-1} - 1} \\ g(x) &= \left(\sqrt{B}\sqrt{\sqrt{K^2-1}-K}\right)x \qquad f(x) = \frac{\operatorname{sn}(g(x),\alpha)}{\sqrt{\sqrt{K^2-1}-K}}. \end{aligned} \tag{30}$$

Then one can show that the pair

$$\begin{aligned} u(x) &= 4\arctan f(x) \\ \tau(x) &= \frac{\sqrt{E}\operatorname{cn}(g(x),\alpha)\operatorname{dn}(g(x),\alpha)}{1 + \frac{\operatorname{sn}^2(g(x),\alpha)}{\sqrt{K^2-1}-K}} \end{aligned} \tag{31}$$

solves system (10). Since $\operatorname{sn}(0,\alpha) = 0$ and $\operatorname{cn}(0,\alpha) = \operatorname{dn}(0,\alpha) = 1$, we see that $u(0) = 0$ and $\tau(0) = \sqrt{E}$. Also, $J(x) = -x^2$, which means that in equation (13) we can conclude that $C = \tau(0)^2 = E$, and that the solution in (16) corresponding to (31) is given by

$$ds^2_{bh} = -\left[\frac{r^2}{l^2} - E\right] dT^2 + \left[\frac{r^2}{l^2} - E\right]^{-1} dr^2, \tag{32}$$

which is a black hole with positive mass E. In this case it is easier to compute the inverse transformation $\Theta = \Psi^{-1}$ in (17) which is simply $l\tau$ for τ in (31).

The string-inspired gravity model, with $V(x) = -\gamma$, $\gamma > 0$, gives rise to the (non-soliton) example

$$u(x) = \frac{A\gamma}{l}x + b, \qquad \tau(x) = -\frac{1}{2A^2\gamma}\cos\left(\frac{A\gamma}{l}x + b\right) + c \tag{33}$$

that solves (10). $J(x) = -\gamma x$ and the choice of $x = -\frac{bl}{A\gamma}$ in (13) gives $C = -\frac{1}{2A^2} + \gamma c$, by which one obtains the solution (see (16))

$$\mathrm{d}s^2_{\mathrm{bh}} = -\left[\frac{\gamma}{l}r + \frac{1}{2A^2} - \gamma c\right]\mathrm{d}T^2 + \left[\frac{\gamma}{l}r + \frac{1}{2A^2} - \gamma c\right]^{-1}\mathrm{d}r^2. \tag{34}$$

$\Psi(T, r)$ in (17) assumes the form

$$\psi_1(T, r) = \frac{l}{A\gamma}\left\{-b + \arccos\left[2A^2\gamma\left(c - \frac{r}{l}\right)\right]\right\} \qquad \psi_2(T, r) = \frac{T - \theta_0}{A}. \tag{35}$$

5. Remarks on non-static solutions

Non-static solutions of the field equations (7) (for string-inspired, spherically symmetric, and Jackiw–Teitelboim (JT) gravity) are given in [2], which complement 2-soliton solutions found in [5] for JT gravity. For example,

Theorem 2. *Given m and $v > 0$ (which we regard as mass and velocity parameters) set $a^2 = 1 + v^2$ and define*

$$u(x, t) = 4\arctan\left[\frac{v\sinh(amx)}{a\cos(vmt)}\right] \qquad \tau(x, t) = \frac{4v^2am\,[\sin(vmt)]\sinh(amx)}{a^2\cos^2(vmt) + v^2\sinh^2(amx)}. \tag{36}$$

Then this pair solves system (7) for $V(x) = -x$, $l = \frac{1}{\sqrt{2}m}$.

As shown in [2], a transformation Ψ is also constructed that takes the metric in (5) to that in (4), for an appropriate value of C. Another application for such transformations Ψ is towards the construction of exact solutions of field equations defined by the Laplacian $\Box^+_{\mathrm{sol}}$ of the metric (5). In the static case at hand, one can prove the following somewhat remarkable commutivity property, which means that Θ and Ψ are transformations of solution spaces:

Theorem 3. *Let $\Box^-_{\mathrm{sol}}$ denote the Laplacian of the metric in equation (20), let $\Box_{\mathrm{bh}}$ denote the Laplacian of the metric in (4) and let $D^{\pm}_{\tau}$ denote the domains*

$$D^{\pm}_{\tau} = \{(x, t) \in \mathbb{R}^2 | \pm\tau'(x) > 0\}. \tag{37}$$

Then for a function $f(T, r)$ and the transformation Θ in (17) (which we have observed is the same as that in (26)), one has

$$\begin{aligned} \Box^+_{\mathrm{sol}}(f\circ\Theta) &= (\Box_{\mathrm{bh}} f)\circ\Theta \qquad \text{on} \quad D^+_{\tau} \\ \Box^-_{\mathrm{sol}}(f\circ\Theta) &= (\Box_{\mathrm{bh}} f)\circ\Theta \qquad \text{on} \quad D^-_{\tau}. \end{aligned} \tag{38}$$

In particular, Θ and Ψ are isometries.

Details of this commutation property are found in [11], in the JT case, where some solutions of the equation $\Box_{\mathrm{sol}}\phi = \mu\phi$ are also presented.

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